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# Derived categories and Morita duality theory

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## Abstract

We define a cotilting bimodule complex as the non-commutative ring version of a dualizing complex, and show that a cotilting bimodule complex includes all indecomposable injective modules in case of Noetherian rings. Moreover we define strong-Morita derived duality, and show that existence of a cotilting bimodule complex is equivalent to one of strong-Morita derived duality. © 1998 Elsevier Science B.V. All rights reserved.

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## 0. Introduction

In algebraic geometry, the notion of dualizing complexes was introduced by Grothendieck and Hartshorne [4], and was studied by several authors. They had started to use technique of local duality, and used developed technique of duality for derived categories [4]. Yekutieli developed this theory to deal with case of non-commutative graded  $k$ -algebras [13]. In representation theory, Rickard gave a ‘Morita theory’ for derived categories of module categories [10]. He also introduced tilting bimodule complexes in case of projective  $k$ -algebras over a commutative ring  $k$ , and studied the relations between tilting bimodule complexes and derived equivalences [11]. Afterwards several authors in representation theory studied derived categories of module categories (for example [5, 7]). We studied cotilting bimodules as the non-commutative ring version of dualizing modules, and the conditions that bimodules induce a localization duality of derived categories [8]. The purpose of this paper is to study a ‘Morita duality theory’ for derived categories in case of coherent rings, that is, the relations between cotilting bimodule complexes and dualities for derived categories. From the point of view of dualizing complexes, this notion is also the non-commutative ring version of dualizing complexes.

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In Section 2, we study bimodule complexes which induce localization dualities of derived categories of modules (Theorem 2.6 and Corollary 2.7), and show that a cotilting bimodule complex induces a Morita derived duality (Corollary 2.8). Moreover, we show a cotilting bimodule complex is a finitely embedding cogenerator, and in case of Noetherian rings, a cotilting bimodule complex includes every injective indecomposable module (Theorem 2.9, Corollaries 2.10, and 2.11). This property is also the non-commutative ring version of residual complexes in algebraic geometry. For an algebra  $A$  over a commutative Noetherian ring  $R$ , we construct a dualizing  $A$ -bimodule complex by using an  $R$ -module dualizing complex (Theorem 2.14 and Corollary 2.15). In Section 3, in case of projective  $k$ -algebras over a commutative ring  $k$ , we give a ‘Morita duality theorem’ for derived categories (Theorem 3.3 and Corollary 3.6). For local rings, we have the uniqueness of the cotilting bimodule complex (Proposition 3.7); as well as the uniqueness of the dualizing complex.

Throughout this paper, we assume that all rings have non-zero unity, and that all modules are unital.

### 1. Preliminaries

Let  $G: \mathcal{U} \rightarrow \mathcal{V}$  and  $F: \mathcal{V} \rightarrow \mathcal{U}$  be contravariant  $\partial$ -functors between triangulated categories. We call  $G$  continuous if  $G$  sends direct sums to direct products (if they exist). We call  $\{G, F\}$  a right adjoint pair if there is a functorial isomorphism  $\text{Hom}_{\mathcal{U}}(X, FY) \cong \text{Hom}_{\mathcal{V}}(Y, GX)$  for all  $X \in \mathcal{U}$  and  $Y \in \mathcal{V}$ . It is easy to see that if  $\{G, F\}$  is a right adjoint pair, then  $G$  and  $F$  are continuous. We call  $\{\mathcal{V}; G, F\}$  a localization duality of  $\mathcal{U}$  provided that  $\{G, F\}$  is a right adjoint pair, and that the natural morphism  $\text{id}_{\mathcal{V}} \rightarrow G \circ F$  is an isomorphism (see [8]).

Let  $\mathcal{A}$  be an additive category,  $K(\mathcal{A})$  a homotopy category of  $\mathcal{A}$ , and  $K^+(\mathcal{A})$ ,  $K^-(\mathcal{A})$  and  $K^b(\mathcal{A})$  full subcategories of  $K(\mathcal{A})$  generated by the bounded below complexes, the bounded above complexes, the bounded complexes, respectively. For a full subcategory  $\mathcal{B}$  of an abelian category  $\mathcal{A}$ , let  $K^{*\text{b}}(\mathcal{B})$  be the full subcategory of  $K^*(\mathcal{B})$  generated by complexes which have bounded homologies, and  $K^*(\mathcal{B})_{\text{Qis}}$  the quotient category of  $K^*(\mathcal{B})$  by the multiplicative set of quasi-isomorphism, where  $*$  = + or  $-$ . We denote  $K^*(\mathcal{A})_{\text{Qis}}$  by  $D^*(\mathcal{A})$ . For a thick abelian subcategory  $\mathcal{C}$  of  $\mathcal{A}$ , we denote by  $D_{\mathcal{C}}^*(\mathcal{A})$  a full subcategory of  $D^*(\mathcal{A})$  generated by complexes of which all homologies belong to  $\mathcal{C}$  (see [4] for details).

For a complex  $X^\bullet := (X^i, d_i)$ , we define the following truncations:

$$\sigma_{>n}(X^\bullet): \dots \rightarrow 0 \rightarrow \text{Im } d_n \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \dots,$$

$$\sigma_{\leq n}(X^\bullet): \dots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow \text{Ker } d_n \rightarrow 0 \rightarrow \dots,$$

$$\tau_{>n}(X^\bullet): \dots \rightarrow 0 \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \dots,$$

$$\tau_{\leq n}(X^\bullet): \dots \rightarrow X^{n+1} \rightarrow X^n \rightarrow 0 \rightarrow \dots.$$

For  $m \leq n$ , we denote by  $K^{[m,n]}(\mathcal{B})$  the full subcategory of  $K(\mathcal{B})$  generated by complexes of the form:  $\dots \rightarrow 0 \rightarrow X^m \rightarrow \dots \rightarrow X^{n+1} \rightarrow X^n \rightarrow 0 \rightarrow \dots$ , and denote by  $D^{[m,n]}(\mathcal{A})$  the full subcategory of  $D(\mathcal{A})$  generated by complexes of which homology  $H^i = 0$  ( $i < m$  or  $n < i$ ).

**2. Cotilting bimodule complexes and Morita derived duality**

For a ring  $A$ , we denote by  $\text{Mod } A$  (resp.,  $A\text{-Mod}$ ) the category of right (resp., left)  $A$ -modules, and denote by  $\text{mod } A$  (resp.,  $A\text{-mod}$ ) the category of finitely presented right (resp., left)  $A$ -modules. We denote by  $\text{Inj } A$  (resp.,  $A\text{-Inj}$ ) the category of injective right (resp., left)  $A$ -modules, and denote by  $\mathcal{P}_A$  (resp.,  ${}_A\mathcal{P}$ ) the category of finitely generated projective right (resp., left) modules. If  $A$  is a right coherent ring, then  $\text{mod } A$  is a thick abelian subcategory of  $\text{Mod } A$ , and then  $D^*(\text{mod } A)$  is equivalent to  $K^{-*}(\mathcal{P}_A)$ . Moreover,  $D^*(\text{mod } A)$  is equivalent to  $D_{\text{mod } A}^*(\text{Mod } A)$ , where  $*$  =  $-$  or  $b$  (see [4]).

For a right  $A$ -module  $U_A$  over a ring  $A$ , we denote by  $\text{add } U_A$  (resp.,  $\text{sum } U_A$ ) the category of right  $A$ -modules which are direct summands of finite direct sums of copies of  $U_A$  (resp., finite direct sums of copies of  $U_A$ ).

For a sequence  $\{X_i^\bullet; f_i: X_i^\bullet \rightarrow X_{i+1}^\bullet\}_{i \geq 1}$  of complexes in  $K(\text{Mod } A)$  (resp.,  $D(\text{Mod } A)$ ), we have the following distinguished triangle in  $K(\text{Mod } A)$  (resp.,  $D(\text{Mod } A)$ ).

$$\bigoplus_i X_i^\bullet \xrightarrow{1\text{-Shift}} \bigoplus_i X_i^\bullet \rightarrow X^\bullet \rightarrow .$$

We denote  $X^\bullet$  by  $\text{hlim}_{i \rightarrow \infty} X_i^\bullet$  and call it the homotopy colimit of the sequence [2].

Similarly, for a sequence  $\{X_i^\bullet; f_i: X_{i+1}^\bullet \rightarrow X_i^\bullet\}_{i \geq 1}$  of complexes in  $K(\text{Mod } A)$  (resp.,  $D(\text{Mod } A)$ ), we have the following distinguished triangle in  $K(\text{Mod } A)$  (resp.,  $D(\text{Mod } A)$ ):

$$X^\bullet \rightarrow \prod_i X_i^\bullet \xrightarrow{1\text{-Shift}} \prod_i X_i^\bullet \rightarrow .$$

We denote  $X^\bullet$  by  $\text{hlim}_{\infty \leftarrow i} X_i^\bullet$ , and call it the homotopy limit of the sequence.

According to [2], for a complex  $X^\bullet \in K(\text{Mod } A)$ , we have the following isomorphisms in  $D(\text{Mod } A)$ :

$$\begin{aligned} \text{hlim}_{n \rightarrow \infty} \tau_{\geq -n} X^\bullet &\cong X^\bullet, & \text{hlim}_{\infty \leftarrow n} \sigma_{> -n} X^\bullet &\cong X^\bullet, \\ \text{hlim}_{n \rightarrow \infty} \sigma_{\leq n} X^\bullet &\cong X^\bullet & \text{and } \text{hlim}_{\infty \leftarrow n} \tau_{\leq n} X^\bullet &\cong X^\bullet. \end{aligned}$$

Spaltenstein, Böskstedt and Neeman defined the triangulated subcategory  $K^s(\text{Inj } A)$  (resp.,  $K^s(\text{Proj } A)$ ) of  $K(\text{Mod } A)$  which consists of special complexes of injective (resp.,

projective) right  $A$ -modules. Given a complex  $X^\bullet \in D(\text{Mod } A)$ , we have the isomorphism  $\text{Hom}_{D(\text{Mod } A)}(X^\bullet, I^\bullet) \cong \text{Hom}_{K(\text{Mod } A)}(X^\bullet, I^\bullet)$  for every complex  $I^\bullet \in K^s(\text{Inj } A)$ . Moreover, for every complex  $X^\bullet \in D(\text{Mod } A)$ , there exists a complex  $I^\bullet \in K^s(\text{Inj } A)$  which has a quasi-isomorphism  $X^\bullet \rightarrow I^\bullet$  in  $K(\text{Mod } A)$ . Similarly, given a complex  $X^\bullet \in D(\text{Mod } A)$ , we have the isomorphism  $\text{Hom}_{D(\text{Mod } A)}(P^\bullet, X^\bullet) \cong \text{Hom}_{K(\text{Mod } A)}(P^\bullet, X^\bullet)$  for every complex  $P^\bullet \in K^s(\text{Inj } A)$ . Moreover, for every complex  $X^\bullet \in D(\text{Mod } A)$ , there exists a complex  $P^\bullet \in K^s(\text{Proj } A)$  which has a quasi-isomorphism  $P^\bullet \rightarrow X^\bullet$  in  $K(\text{Mod } A)$  (see [2, 12] for details).

Let  $A$  and  $B$  be rings. A complex  $X^\bullet = (X^i; d_i: X^i \rightarrow X^{i+1})$  is called a  $B$ - $A$ -bimodule complex provided that all  $X^i$  are  $B$ - $A$ -bimodules and all  $d_i$  are  $B$ - $A$ -bimodule morphisms.

**Definitions.** Let  $A$  be a right coherent ring and  $B$  a left coherent ring. A  $B$ - $A$ -bimodule complex  ${}_B U_A^\bullet$  is called a cotilting  $B$ - $A$ -bimodule complex provided that it satisfies the following:

(C1)  ${}_B U_A^\bullet$  is contained in  $D_{\text{mod } A}^b(\text{Mod } A)$  as a right  $A$ -module complex, and is contained in  $D_{B\text{-mod}}^b(B\text{-Mod})$  as a left  $B$ -module complex.

(C2r)  ${}_B U_A^\bullet$  belongs to  $K^b(\text{Inj } A)$  as a right  $A$ -module complex;

(C2l)  ${}_B U_A^\bullet$  belongs to  $K^b(B\text{-Inj})$  as a left  $B$ -module complex;

(C3r)  $\text{Hom}_{D(\text{Mod } A)}(U^\bullet, U^\bullet[i]) = 0$  for all  $i \neq 0$ ;

(C3l)  $\text{Hom}_{D(B\text{-Mod})}(U^\bullet, U^\bullet[i]) = 0$  for all  $i \neq 0$ ;

(C4r) the natural left multiplication morphism  $B \rightarrow \text{Hom}_{D(\text{Mod } A)}({}_B U_A^\bullet, {}_B U_A^\bullet)$  is a ring isomorphism;

(C4l) the natural right multiplication morphism  $A^{\text{op}} \rightarrow \text{Hom}_{D(B\text{-Mod})}({}_B U_A^\bullet, {}_B U_A^\bullet)$  is a ring isomorphism.

In case of  $B = A$ , we will call a cotilting  $A$ - $A$ -bimodule complex a dualizing  $A$ -bimodule complex.

We say that  $A$  is a left Morita (resp., strong-Morita) derived dual of  $B$  if there exist contravariant continuous  $\delta$ -functors  $F: D(\text{Mod } A) \rightarrow D(B\text{-Mod})$  and  $G: D(B\text{-Mod}) \rightarrow D(\text{Mod } A)$  which satisfy the condition (D1) (resp., the conditions (D1), (D2r) and (D2l)):

(D1)  $F$  and  $G$  induce a duality between  $D_{\text{mod } A}^b(\text{Mod } A)$  and  $D_{B\text{-mod}}^b(B\text{-Mod})$ ;

(D2r) the image of  $F|_{D^b(\text{Mod } A)}$  is contained in  $D^b(B\text{-Mod})$ ;

(D2l) the image of  $G|_{D^b(B\text{-Mod})}$  is contained in  $D^b(\text{Mod } A)$ .

**Remark.** Let  $F: D(\text{Mod } A) \rightarrow D(B\text{-Mod})$  and  $G: D(B\text{-Mod}) \rightarrow D(\text{Mod } A)$  be  $\delta$ -functors satisfying that  $A$  is a left Morita derived dual of  $B$ . Then  $\{F, G\}$  is a right adjoint pair as functors between  $D_{\text{mod } A}^b(\text{Mod } A)$  and  $D_{B\text{-mod}}^b(B\text{-Mod})$ . It need not be a right adjoint pair as functors between  $D(\text{Mod } A)$  and  $D(B\text{-Mod})$ , but we have the following statement.

**Proposition 2.1.** *Let  $A$  be a right coherent ring,  $B$  a left coherent rings (resp., a left Noetherian ring), and let  $F: D(\text{Mod } A) \rightarrow D(B\text{-Mod})$  and  $G': D(B\text{-Mod}) \rightarrow D(\text{Mod } A)$*

be contravariant continuous  $\partial$ -functors satisfying that  $A$  is a left Morita (resp., strong-Morita) derived dual of  $B$ . Then there exists a  $\partial$ -functor  $G: D(B\text{-Mod}) \rightarrow D(\text{Mod } A)$  which satisfies the following:

- (a)  $\{F, G\}$  is a right adjoint pair as functors between  $D(\text{Mod } A)$  and  $D(B\text{-Mod})$ .
- (b)  $\{F, G\}$  induces that  $A$  is a left Morita (resp., strong-Morita) derived dual of  $B$ .

**Proof.** According to [9, Theorems 3.1 and 4.1], there exists a  $\partial$ -functor  $G: D(B\text{-Mod}) \rightarrow D(\text{Mod } A)$  such that  $\{F, G\}$  is a right adjoint pair as functors between  $D(\text{Mod } A)$  and  $D(B\text{Mod})$ . By the above remark, for every complex  $Y^\bullet$  in  $D_{B\text{-mod}}^b(B\text{-Mod})$ , we have the following isomorphisms:

$$\begin{aligned} H^i(GY^\bullet) &\cong \text{Hom}_{D(\text{Mod } A)}(A, GY^\bullet[i]) \cong \text{Hom}_{D(B\text{-Mod})}(Y^\bullet, FA[i]) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(A, G'Y^\bullet[i]) \cong H^i(G'Y^\bullet) \quad \text{for all } i. \end{aligned}$$

Then  $GY^\bullet$  belongs to  $D_{\text{mod } A}^b(\text{Mod } A)$ . Hence, by adjointness of  $F, G$  is isomorphic to  $G'$  as functors from  $D_{B\text{-mod}}^b(B\text{-Mod})$  to  $D_{\text{mod } A}^b(\text{Mod } A)$ . Let  $lI(B)$  be the set of left ideals of  $B$ . In case of  $B$  being left Noetherian, there exists some integer  $n$  such that we have the following isomorphisms:

$$\begin{aligned} \text{Hom}_{D(B\text{-Mod})} \left( \bigoplus_{J \in lI(B)} B/J, FA[i] \right) &\cong \prod_{J \in lI(B)} \text{Hom}_{D(B\text{-Mod})}(B/J, FA[i]) \\ &\cong \prod_{J \in lI(B)} \text{Hom}_{D(\text{Mod } A)}(A, G'B/J[i]) \\ &\cong \text{Hom}_{D(\text{Mod } A)} \left( A, \prod_{J \in lI(B)} G'B/J[i] \right) \\ &\cong \text{Hom}_{D(\text{Mod } A)} \left( A, G' \left( \bigoplus_{J \in lI(B)} B/J \right) [i] \right) \\ &= 0 \quad \text{for all } i > n. \end{aligned}$$

By Lemma 3.1(a), we get  $FA \in D^b(B\text{-Mod})_{\text{fid}}$ . For every complex  $Y^\bullet$  in  $D^b(B\text{-Mod})$ , there exist integers  $m \leq n$  such that we have the following isomorphisms:

$$\begin{aligned} H^i(GY^\bullet) &\cong \text{Hom}_{D(\text{Mod } A)}(A, GY^\bullet[i]) \\ &= \text{Hom}_{D(B\text{-Mod})}(Y^\bullet, FA[i]) \\ &= 0 \quad \text{for all } i < m \text{ or } i > n. \end{aligned}$$

Then  $GY^\bullet$  belongs to  $D^b(\text{Mod } A)$ , and therefore the image of  $G|_{D^b(B\text{-Mod})}$  is contained in  $D^b(\text{Mod } A)$ . Hence  $F$  and  $G$  induce that  $A$  is a left strong-Morita derived dual of  $B$ .  $\square$

**Lemma 2.2.** *Let  $A$  and  $B$  be rings,  ${}_B U_A$  a  $B$ - $A$ -bimodule. Then  $\{\text{Hom}_A(-, {}_B U_A): \text{Mod } A \rightarrow B\text{-Mod}, \text{Hom}_B(-, {}_B U_A): B\text{-Mod} \rightarrow \text{Mod } A\}$  is a right adjoint pair.*

**Lemma 2.3.** *Let  ${}_B U_A^\bullet$  be a  $B$ - $A$ -bimodule complex. Then  $\{\mathbf{R} \text{Hom}_A^\bullet(-, {}_B U_A^\bullet) : D(\text{Mod } A) \rightarrow D(B\text{-Mod}), \mathbf{R} \text{Hom}_B^\bullet(-, {}_B U_A^\bullet) : D(B\text{-Mod}) \rightarrow D(\text{Mod } A)\}$  is a right adjoint pair.*

**Proof.** According to [2], for a complex  $X_A^\bullet \in D(\text{Mod } A)$ , there exist complex  $P^\bullet \in K^s(\text{Proj } A)$  such that  $X^\bullet$  is isomorphic to  $P^\bullet$  in  $D(\text{Mod } A)$ . Similarly, for a complex  ${}_B Y^\bullet \in D(B\text{-Mod})$ , there exists a complex  $Q^\bullet \in K^s(B\text{-Proj})$  such that  $Y^\bullet$  is isomorphic to  $Q^\bullet$  in  $D(B\text{-Mod})$ . Then we have the following isomorphisms:

$$\begin{aligned} \text{Hom}_{D(\text{Mod } A)}(X^\bullet, \mathbf{R} \text{Hom}_B^\bullet(Y^\bullet, {}_B U_A^\bullet)) &\cong H^0 \text{Hom}_A^\bullet(P^\bullet, \text{Hom}_B^\bullet(Q^\bullet, {}_B U_A^\bullet)) \\ &\cong H^0 \text{Hom}_B^\bullet(Q^\bullet, \text{Hom}_A^\bullet(P^\bullet, {}_B U_A^\bullet)) \\ &\cong \text{Hom}_{D(B\text{-Mod})}(Y^\bullet, \mathbf{R} \text{Hom}_A^\bullet(X^\bullet, {}_B U_A^\bullet)). \quad \square \end{aligned}$$

**Definition.** Let  $\mathcal{U}$  be a family of objects of  $D^{[m,n]}(\text{Mod } A)$ . We call a complex  $X^\bullet$  in  $D(\text{Mod } A)$  a  $\mathcal{U}$ -limit complex with  $(\{X_i^\bullet\}_{i \geq 0}; r)$  if there exist an integer  $r$  and a sequence of the following distinguished triangles:

$$\begin{aligned} U_1^\bullet[-1] \rightarrow X_1^\bullet \rightarrow X_0^\bullet \rightarrow, \\ U_2^\bullet[-2] \rightarrow X_2^\bullet \rightarrow X_1^\bullet \rightarrow, \\ \dots \\ U_n^\bullet[-n] \rightarrow X_n^\bullet \rightarrow X_{n-1}^\bullet \rightarrow, \\ \dots, \end{aligned}$$

where  $X_0^\bullet$  and  $U_i^\bullet$  belong to  $\mathcal{U}[r]$  for all  $i \geq 1$ , such that  $X^\bullet$  is isomorphic to  $\text{hlim}_{\infty \leftarrow i} X_i^\bullet$  in  $D(\text{Mod } A)$ . In case of  $\mathcal{U} = \text{add } U_A^\bullet$  for some complex  $U_A^\bullet$  of  $D^{[m,n]}(\text{Mod } A)$ , we simply call a  $\mathcal{U}$ -limit complex a  $U_A^\bullet$ -limit complex.

**Lemma 2.4.** *Let  $\mathcal{U}$  be a family of objects of  $D^{[m,n]}(\text{Mod } A)$ . For a  $\mathcal{U}$ -limit complex  $X^\bullet$  with  $(\{X_i^\bullet\}_{i \geq 0}; r)$ , the following hold:*

- (a) *We have  $X_k^\bullet \in D^{[s,t+k]}(\text{Mod } A)$  for all  $k \geq 0$ , where  $s = m - r$  and  $t = n - r$ .*
- (b) *We have an isomorphism  $\sigma_{\leq s+k-2} X_k^\bullet \cong \sigma_{\leq s+k-2} X_{k-1}^\bullet$  in  $D(\text{Mod } A)$  for every  $k \geq 1$ , where  $s = m - r$ .*
- (c) *If  $A$  is a right coherent ring, and if  $\mathcal{U}$  is a family of objects of  $D_{\text{mod } A}^{[m,n]}(\text{Mod } A)$ , then  $X^\bullet$  belongs to  $D_{\text{mod } A}^+(\text{Mod } A)$ .*

**Proof.** It is straightforward.  $\square$

**Lemma 2.5.** *Let  ${}_B U_A^\bullet$  be a  $B$ - $A$ -bimodule complex satisfying the conditions (C1) and (C2r), and  $\mathcal{U}$  a family of complexes in  $D^{[m,n]}(\text{Mod } A)$ . If  $X^\bullet$  is a  $\mathcal{U}$ -limit complex with a sequence  $\{X_i^\bullet\}_{i \geq 0}$ , then the induced natural morphism  $\text{hlim}_{i \rightarrow \infty} \text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet) \rightarrow \text{Hom}_A^\bullet(\text{hlim}_{\infty \leftarrow i} X_i^\bullet, {}_B U_A^\bullet)$  is an isomorphism in  $D(B\text{-Mod})$ .*

**Proof.** It is easy to see that we have the following commutative diagram in  $D(B\text{-Mod})$ :

$$\begin{array}{ccc} \text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet) = \text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet) & & \\ \downarrow & & \downarrow \\ \text{hlim}_{i \rightarrow \infty} \text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet) \xrightarrow{\eta} & \text{Hom}_A^\bullet & \left( \text{hlim}_{\infty \leftarrow i} X_i^\bullet, {}_B U_A^\bullet \right). \end{array}$$

We may assume  ${}_B U_A^\bullet$  is contained in  $K^{[0, t]}(\text{Inj } A)$ . Given an integer  $k$ , we have the following isomorphisms:

$$\begin{aligned} H^k \text{Hom}_A^\bullet \left( \text{hlim}_{\infty \leftarrow i} X_i^\bullet, {}_B U_A^\bullet \right) &\cong \text{Hom}_{D(\text{Mod } A)} \left( \text{hlim}_{\infty \leftarrow i} X_i^\bullet, {}_B U_A^\bullet[k] \right) \\ &\cong \text{Hom}_{D(\text{Mod } A)} \left( \sigma_{\leq t-k} \text{hlim}_{\infty \leftarrow i} X_i^\bullet, {}_B U_A^\bullet[k] \right) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(\sigma_{\leq t-k} X_p^\bullet, {}_B U_A^\bullet[k]) \quad \text{for some } p \gg 0 \\ &\cong \text{Hom}_{D(\text{Mod } A)}(X_p^\bullet, {}_B U_A^\bullet[k]) \\ &\cong H^k \text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet). \end{aligned}$$

Moreover, there exists an integer  $q$  such that we have the following isomorphisms for all  $j \geq 0$ :

$$\begin{aligned} H^k \text{Hom}_A^\bullet(X_q^\bullet, {}_B U_A^\bullet) &\cong \text{Hom}_{D(\text{Mod } A)}(X_q^\bullet, {}_B U_A^\bullet[k]) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(\sigma_{\leq t-k} X_q^\bullet, {}_B U_A^\bullet[k]) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(\sigma_{\leq t-k} X_{q+j}^\bullet, {}_B U_A^\bullet[k]) \\ &\cong H^k \text{Hom}_A^\bullet(X_{q+j}^\bullet, {}_B U_A^\bullet). \end{aligned}$$

Then we have the isomorphism  $H^k \text{Hom}_A^\bullet(X_q^\bullet, {}_B U_A^\bullet) \cong H^k \text{hlim}_{i \rightarrow \infty} \text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet)$ . For all integers  $r \geq \max(p, q)$ , we have the following commutative diagram:

$$\begin{array}{ccc} H^k \text{Hom}_A^\bullet(X_r^\bullet, {}_B U_A^\bullet) = H^k \text{Hom}_A^\bullet(X_r^\bullet, {}_B U_A^\bullet) & & \\ \downarrow & & \downarrow \\ H^k \text{hlim}_{i \rightarrow \infty} \text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet) \xrightarrow{H^k \eta} & H^k \text{Hom}_A^\bullet & \left( \text{hlim}_{\infty \leftarrow i} X_i^\bullet, {}_B U_A^\bullet \right), \end{array}$$

where vertical arrows are isomorphisms. Therefore  $H^k \eta$  is an isomorphism, and hence  $\eta$  is an isomorphism in  $D(B\text{-Mod})$ .  $\square$

**Theorem 2.6.** *Let  $A$  be a right coherent ring,  $B$  a left coherent ring,  ${}_B U_A^\bullet$  a  $B$ - $A$ -bimodule complex satisfying the conditions (C1), (C2r), (C3r) and (C4r). Then  $\{D_{B\text{-mod}}^-(B\text{-Mod}); \text{Hom}_A^\bullet(-, {}_B U_A^\bullet), R^- \text{Hom}_B^\bullet(-, {}_B U_A^\bullet)\}$  is a localization duality of  $D_{\text{mod } A}^+(\text{Mod } A)$ , and the image of  $\text{Hom}_A^\bullet(-, {}_B U_A^\bullet)|_{D_{\text{mod } A}^+(\text{Mod } A)}$  is contained in  $D_{B\text{-mod}}^b(B\text{-Mod})$ . Moreover, every complex in  $D_{\text{mod } A}^+(\text{Mod } A)$  is a  $U_A^\bullet$ -limit complex if and only if  $\text{Hom}_A^\bullet(-, {}_B U_A^\bullet)$  and  $R^- \text{Hom}_B^\bullet(-, {}_B U_A^\bullet)$  induce the duality between  $D_{B\text{-mod}}^-(B\text{-Mod})$  and  $D_{\text{mod } A}^+(\text{Mod } A)$ .*

**Proof.** The condition (C2r) implies the existence of  $R^+ \text{Hom}_B^\bullet(-, {}_B U_A^\bullet) \cong \text{Hom}_A^\bullet(-, {}_B U_A^\bullet) : D^+(\text{Mod } A) \rightarrow D^-(B\text{-Mod})$ . We have  $\text{Hom}_A^\bullet(P_A, {}_B U_A^\bullet)$  belongs to  $\text{add } {}_B U^\bullet$  for all  $P \in \mathcal{P}_A$ . Then, according to [4, Ch. 1, Proposition 7.3], we can consider  $R^+ \text{Hom}_A^\bullet(-, {}_B U_A^\bullet) : D_{\text{mod } A}^+(\text{Mod } A) \rightarrow D_{B\text{-mod}}^-(B\text{-Mod})$ , and the image of  $R^+ \text{Hom}_A^\bullet(-, {}_B U_A^\bullet)|_{D_{\text{mod } A}^+(\text{Mod } A)}$  is contained in  $D_{B\text{-mod}}^b(B\text{-Mod})$ . It is clear that  $R^+ \text{Hom}_B^\bullet(-, {}_B U_A^\bullet) : D_{B\text{-mod}}^-(B\text{-Mod}) \rightarrow D^\bullet(\text{Mod } A)$  exists. Since  $D_{B\text{-mod}}^-(B\text{-Mod})$  is equivalent to  $D^-(B\text{-Mod})$ ,  $D_{B\text{-mod}}^-(B\text{-Mod})$  is equivalent to  $K^-(B\mathcal{P})$ . Given a complex  $X^\bullet \in D_{B\text{-mod}}^-(B\text{-Mod})$ , there exists a complex  $P^\bullet \in K^-(B\mathcal{P})$  such that  $X^\bullet$  is isomorphic to  $P^\bullet$  in  $D_{B\text{-mod}}^-(B\text{-Mod})$ . Since  $R^- \text{Hom}_B^\bullet(-, {}_B U_A^\bullet)$  is isomorphic to  $R^- \text{Hom}_B^\bullet(\text{hlim}_{n \rightarrow \infty} \tau_{\geq -n} P^\bullet, {}_B U_A^\bullet) \cong \text{hlim}_{\infty \leftarrow n} R^- \text{Hom}_B^\bullet(\tau_{\geq -n} P^\bullet, {}_B U_A^\bullet)$ ,  $R^- \text{Hom}_B^\bullet(P^\bullet, {}_B U_A^\bullet)$  is a  $U_A^\bullet$ -limit complex. By Lemma 2.4,  $R^- \text{Hom}_B^\bullet(P^\bullet, {}_B U_A^\bullet)$  is contained in  $D_{\text{mod } A}^+(\text{Mod } A)$ . Also, the condition (C3r) implies that the natural morphism  $\tau_{\geq -n} P^\bullet \rightarrow R^+ \text{Hom}_A^\bullet(R^- \text{Hom}_B^\bullet(\tau_{\geq -n} P^\bullet, {}_B U_A^\bullet), {}_B U_A^\bullet)$  is an isomorphism in  $D(B\text{-Mod})$ . Therefore, according to Lemma 2.5, we have the following commutative diagram in  $D(B\text{-Mod})$ :

$$\begin{array}{ccc}
 \text{hlim}_{n \rightarrow \infty} \tau_{\geq -n} P^\bullet & \rightarrow & \text{hlim}_{n \rightarrow \infty} R^+ \text{Hom}_A^\bullet(R^- \text{Hom}_B^\bullet(\tau_{\geq -n} P^\bullet, {}_B U_A^\bullet), {}_B U_A^\bullet) \\
 \parallel & & \downarrow \\
 & & R^+ \text{Hom}_A^\bullet\left(\text{hlim}_{\infty \leftarrow n} R^- \text{Hom}_B^\bullet(\tau_{\geq -n} P^\bullet, {}_B U_A^\bullet), {}_B U_A^\bullet\right) \\
 & & \downarrow \\
 \text{hlim}_{n \rightarrow \infty} \tau_{\geq -n} P^\bullet & \rightarrow & R^+ \text{Hom}_A^\bullet\left(R^- \text{Hom}_B^\bullet\left(\text{hlim}_{n \rightarrow \infty} \tau_{\geq -n} P^\bullet, {}_B U_A^\bullet\right), {}_B U_A^\bullet\right) \\
 \downarrow & & \downarrow \\
 P^\bullet & \longrightarrow & R^+ \text{Hom}_A^\bullet(R^- \text{Hom}_B^\bullet(P, {}_B U_A^\bullet), {}_B U_A^\bullet),
 \end{array}$$

where vertical arrows are isomorphisms in  $D(B\text{-Mod})$ . Hence  $P^\bullet \rightarrow R^+ \text{Hom}_A^\bullet(R^- \text{Hom}_B^\bullet(P^\bullet, {}_B U_A^\bullet), {}_B U_A^\bullet)$  is an isomorphism in  $D(B\text{-Mod})$ .

By the above, it is easy to see if  $\text{Hom}_A^\bullet(-, {}_B U_A^\bullet)$  and  $\text{Hom}_B^\bullet(-, {}_B U_A^\bullet)$  induce the duality between  $D_{B\text{-mod}}^-(B\text{-Mod})$  and  $D_{\text{mod } A}^+(\text{Mod } A)$ , then every complex in  $D_{\text{mod } A}^+(\text{Mod } A)$  is a  $U_A^\bullet$ -limit complex. Conversely, if every complex  $X^\bullet$  in  $D_{\text{mod } A}^+(\text{Mod } A)$  is a  $U_A^\bullet$ -limit complex, then there exist an integer  $r$  and a sequence of the following distinguished triangles:

$$\begin{array}{l}
 U_1^\bullet[-1] \rightarrow X_1^\bullet \rightarrow X_0^\bullet \rightarrow, \\
 U_2^\bullet[-2] \rightarrow X_2^\bullet \rightarrow X_1^\bullet \rightarrow, \\
 \dots \\
 U_n^\bullet[-n] \rightarrow X_n^\bullet \rightarrow X_{n-1}^\bullet \rightarrow, \\
 \dots,
 \end{array}$$



where  $X_0^\bullet$  and  $U_i^\bullet$  belong to  $(\text{add } U_A^\bullet)[r]$  for all  $i \geq 1$ , such that  $X^\bullet$  is isomorphic to  $\text{hlim}_{\infty \leftarrow i} X_i^\bullet$  in  $D(\text{Mod } A)$ . Since  $U_i^\bullet[-i] \rightarrow \text{Hom}_B^\bullet(\text{Hom}_A^\bullet(U_i^\bullet[-i], {}_B U_A^\bullet), {}_B U_A^\bullet)$  is an isomorphism in  $D(\text{Mod } A)$  for all  $i$ , the natural morphism  $X_i^\bullet \rightarrow \text{Hom}_B^\bullet(\text{Hom}_A^\bullet(X_i^\bullet, {}_B U_A^\bullet), {}_B U_A^\bullet)$  is an isomorphism in  $D(\text{Mod } A)$  for all  $i$ . By Lemma 2.5, the natural morphism  $\text{hlim}_{\infty \leftarrow i} X_i^\bullet \rightarrow \text{Hom}_B^\bullet(\text{Hom}_A^\bullet(\text{hlim}_{\infty \leftarrow i} X_i^\bullet, {}_B U_A^\bullet), {}_B U_A^\bullet)$  is an isomorphism in  $D(\text{Mod } A)$ . Therefore,  $\text{Hom}_B^\bullet(-, {}_B U_A^\bullet): D_{B\text{-mod}}^-(B\text{-Mod}) \rightarrow D_{\text{mod } A}^+(\text{Mod } A)$  is dense, and hence  $\text{Hom}_A^\bullet(-, {}_B U_A^\bullet)$  and  $\text{Hom}_B^\bullet(-, {}_B U_A^\bullet)$  induce the duality between  $D_{B\text{-mod}}^-(B\text{-Mod})$  and  $D_{\text{mod } A}^+(\text{Mod } A)$ .  $\square$

**Corollary 2.7.** *Let  $A$  be right coherent ring,  $B$  a left coherent ring,  ${}_B U_A^\bullet$  a  $B$ - $A$ -bimodule complex satisfying the conditions (C1), (C2r), (C2l), (C3r) and (C4r). Then  $\{D_{B\text{-mod}}^b(B\text{-Mod}); \text{Hom}_A^\bullet(-, {}_B U_A^\bullet), \text{Hom}_B^\bullet(-, {}_B U_A^\bullet)\}$  is a localization duality of  $D_{B\text{-mod}}^b(\text{Mod } A)$ .*

**Proof.** By the condition (C2l), it is easy to see that the image of  $R^- \text{Hom}_B^\bullet(-, {}_B U_A^\bullet) \cong \text{Hom}_A^\bullet(-, {}_B U_A^\bullet)$  is contained in  $D_{\text{mod } A}^p(\text{Mod } A)$ . We are done by Theorem 2.6.  $\square$

**Corollary 2.8.** *Let  $A$  be a right coherent ring,  $B$  a left coherent ring,  ${}_B U_A^\bullet$  a cotilting  $B$ - $A$ -bimodule complex. Then  $A$  is a left strong-Morita derived dual of  $B$ , and there is a duality between  $D_{\text{mod } A}(\text{Mod } A)$  and  $D_{B\text{-mod}}(B\text{-Mod})$ .*

**Proof.** It is clear that  $\text{Hom}_A^\bullet(-, {}_B U_A^\bullet)$  and  $\text{Hom}_B^\bullet(-, {}_B U_A^\bullet)$  are continuous  $\partial$ -functors. According to Lemma 2.3 and Corollary 2.7,  $A$  is a left strong-Morita derived dual of  $B$ . Since  $\text{Hom}_A^\bullet(-, {}_B U_A^\bullet)$  and  $\text{Hom}_B^\bullet(-, {}_B U_A^\bullet)$  are way-out in both directions, by [4, Ch. 1, Proposition 7.1], we deduce the assertion.  $\square$

Let  $\mathcal{A}$  be an abelian category,  $\mathcal{B}$  a full subcategory of  $\mathcal{A}$ . We call an object  $X \in \mathcal{A}$  a finitely embedding cogenerator for  $\mathcal{B}$  provided that every object in  $\mathcal{B}$  has an injection to some finite direct sum of copies of  $X$  in  $\mathcal{A}$ .

**Theorem 2.9.** *Let  $A$  be a right coherent ring,  $B$  a left coherent ring, and  ${}_B U_A^\bullet$  a  $B$ - $A$ -bimodule complex which satisfies the conditions (C1) and (C2r). Assume that the image of  $\text{Hom}_A^\bullet(-, {}_B U_A^\bullet): D_{\text{mod } A}^-(\text{Mod } A) \rightarrow D_{B\text{-mod}}^+(\text{B-Mod})$  contains  $B\text{-mod}$ . If  $E^\bullet$  is a complex  $E^{-s} \rightarrow \dots \rightarrow E^0 \rightarrow \dots$  in  $K^+(\text{B-Inj})$  which is isomorphic to  ${}_B U_A^\bullet$  in  $D(B\text{-Mod})$ , then  $\bigoplus_{i \geq -s} E^i$  is a finitely embedding cogenerator for  $B\text{-mod}$ , and then  $\prod_{i \geq -s} E^i$  is a finitely embedding injective cogenerator for  $B\text{-mod}$ .*

**Proof.** Since  $D_{\text{mod } A}^-(\text{Mod } A)$  is equivalent to  $D^-(\text{mod } A)$ ,  $D_{\text{mod } A}^-(\text{Mod } A)$  is equivalent to  $K^-(\text{sum } A_A)$ . By assumption, for every  $X \in B\text{-mod}$ , there exists a complex  $P^\bullet$  in  $K^-(\text{sum } A_A)$  such that  $\text{Hom}_A^\bullet(P^\bullet, {}_B U_A^\bullet)$  is isomorphic to  $X$  in  $D^b(B\text{-Mod})$ . Since  $\text{Hom}_A^\bullet(P^\bullet, {}_B U_A^\bullet)$  is a  ${}_B U^\bullet$ -limit complex, there exists an integer  $n$  such that we have isomorphisms  $H^i \text{Hom}_A^\bullet(P^\bullet, {}_B U_A^\bullet) \cong H^i \text{Hom}_A^\bullet(\tau_{\geq n} P^\bullet, {}_B U_A^\bullet)$  for all  $i \leq 0$ . We may assume  $\tau_{\geq n} P^\bullet$  is a complex  $P^n \rightarrow \dots \rightarrow P^m$ , where  $P^i \in \text{sum } A_A$  ( $n \leq i \leq m$ ).

Then  $\text{Hom}_A^\bullet(P^i, {}_B U_A^\bullet)$  is isomorphic to  $E_i^\bullet$  for some  $E_i^\bullet \in \text{sum } E^\bullet$  ( $n \leq i \leq m$ ). Therefore  $\text{Hom}_A^\bullet(\tau_{\geq n} P^\bullet, {}_B U_A^\bullet)$  is isomorphic to a iterated mapping cone complex  $E_n^\bullet[n] \oplus E_{n+1}^\bullet[n+1] \oplus \dots \oplus E_m^\bullet[m]$ . The complex  $E_n^\bullet[n] \oplus E_{n+1}^\bullet[n+1] \oplus \dots \oplus E_m^\bullet[m]$  is of the form  $I^{-m-s} \rightarrow \dots \rightarrow I^{-1} \rightarrow I^0 \rightarrow \dots$ , where  $I^j \in \text{add}(\bigoplus_{i \geq -s} E^i)$ . Then we have the following exact sequences:

$$0 \rightarrow I^{-m-s} \rightarrow \dots \rightarrow I^{-1} \rightarrow \text{Im } d_{-1} \rightarrow 0 \tag{1}$$

$$0 \rightarrow \text{Im } d_{-1} \rightarrow \text{Ker } d_0 \rightarrow X \rightarrow 0. \tag{2}$$

Since  $I^i$  is injective ( $-s \leq i \leq -1$ ),  $\text{Im } d_{-1}$  is injective. Therefore, the exact sequence (2) splits, and hence  $X$  has an injection to  $I^0$ . By  $I^0 \in \text{add}(\bigoplus_{i \geq -s} E^i)$ ,  $X$  is embedded in some finite direct sum of copies of  $\bigoplus_{i \geq -s} E^i$ .  $\square$

**Corollary 2.10.** *Let  $A$  be a right coherent ring,  $B$  a left coherent ring  ${}_B U_A^\bullet$  a  $B$ - $A$ -bimodule complex  ${}_B U_A^0 \rightarrow \dots \rightarrow {}_B U_A^n$  satisfying the conditions (C1), (C2r), (C2l), (C3r) and (C4r). Then  $\bigoplus_{i=0}^n U^i$  is a finitely embedding injective cogenerator for  $B\text{-mod}$ .*

**Proof.** By Corollary 2.7 and Theorem 2.9.  $\square$

**Corollary 2.11.** *Let  $A$  be a right coherent ring,  $B$  a left coherent ring,  ${}_B U_A^\bullet$  a cotilting  $B$ - $A$ -bimodule complex  ${}_B U_A^0 \rightarrow \dots \rightarrow {}_B U_A^n$ . Then  $\bigoplus_{i=0}^n U^i$  is a finitely embedding injective cogenerator for  $B\text{-mod}$ .*

**Corollary 2.12.** *Let  $A$  be a right coherent ring,  $B$  a left Noetherian ring,  ${}_B U_A^\bullet$  a cotilting  $B$ - $A$ -bimodule complex  ${}_B U_A^0 \rightarrow \dots \rightarrow {}_B U_A^n$ . Then every injective indecomposable left  $B$ -module is isomorphic to a direct summand of some  ${}_B U_A^i$ .*

**Lemma 2.13.** *Let  $A, B, C$  and  $D$  be rings,  ${}_B X_A^\bullet$  a bounded above  $A$ - $B$ -bimodule complex which is contained in  $K^-(\mathcal{P}_B)$ ,  ${}_C Y_A^\bullet$  a bounded below  $C$ - $B$ -bimodule complex, and  ${}_C Z_D^\bullet$  a bounded  $C$ - $D$ -bimodule complex. Then we have the natural  $A$ - $D$ -bimodule complex isomorphism  ${}_A X_B^\bullet \hat{\otimes}_B \text{Hom}_C^\bullet({}_C Y_B^\bullet, {}_C Z_D^\bullet) \cong \text{Hom}_C^\bullet(\text{Hom}_B^\bullet({}_A X_B^\bullet, {}_C Y_B^\bullet), {}_C Z_D^\bullet)$ .*

**Proof.** Let  $X$  be a  $A$ - $B$ -bimodule which is finitely generated projective as a right  $B$ -module,  $Y$  a  $C$ - $B$ -bimodule, and  $Z$  a  $C$ - $D$ -bimodule. Then we have the natural  $A$ - $D$ -bimodule isomorphism  $X \otimes_B \text{Hom}_C(Y, Z) \rightarrow \text{Hom}_C(\text{Hom}_B(X, Y), Z)$  by elementary correspondence ( $x \otimes f \mapsto (g \mapsto f(g(x)))$ ). Then we clearly get the statement.  $\square$

Following Rickard [11], we call an  $A$ - $B$ -bimodule complex  ${}_A T_B^\bullet$  a tilting  $A$ - $B$ -bimodule complex if it satisfies the conditions (C3r), (C3l), (C4r), (C4l) and

(T1)  ${}_A T_B^\bullet$  belongs to  $K^b(\mathcal{P}_B)$  as a right  $B$ -module complex, and belongs to  $K^b({}_A \mathcal{P})$  as a left  $A$ -module complex.

In case of finite dimensional  $k$ -algebras over a field  $k$ , we defined a cotilting module complex by using a duality  $\text{Hom}_k(-, k): \text{mod } A \rightarrow A\text{-mod}$  [7]. We construct a cotilting bimodule complex by using dualizing complexes.

**Theorem 2.14.** *Let  $R$  be a commutative Noetherian ring with a dualizing complex  $\omega^\bullet$ ,  $A$  and  $B$   $R$ -algebras which are finitely generated  $R$ -modules. If  ${}_A T_B^\bullet$  is a tilting  $A$ - $B$ -bimodule complex, then  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  is a cotilting  $B$ - $A$ -bimodule complex.*

**Proof.** It is clear that  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  is contained in  $K^b(\text{Inj } A)$  as a right  $A$ -module complex, and is contained in  $K^b(B\text{-Inj})$  as a left  $B$ -module complex. Since  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  is an  $\omega^\bullet$ -limit complex with a sequence  $\{\text{Hom}_R(\tau_{\geq -n} {}_A T_B^\bullet, \omega^\bullet)\}$ ,  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  is contained in  $D_{\text{mod } R}^b(\text{Mod } R)$ . Since  $A$  and  $B$  are finitely generated  $R$ -modules, every homology of  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  is a finitely generated  $R$ -module, and hence finitely generated as a right  $A$ -module and as a left  $B$ -module. Therefore,  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  is contained in  $D_{\text{mod } A}^b(\text{Mod } A)$  as a right  $A$ -module complex, and is contained in  $D_{B\text{-mod}}^b(B\text{-Mod})$  as a left  $B$ -module complex. In order that  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  satisfies the conditions (C3r) and (C4r), it suffices to show that the natural morphism  $A_A \rightarrow R^b \text{Hom}_B(R^b \text{Hom}_A(A_A, \text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)), \text{Hom}_R({}_A T_B^\bullet, \omega^\bullet))$  is an isomorphism in  $D_{\text{mod } A}^b(\text{Mod } A)$ . By Lemma 2.13, we have the following isomorphisms in  $D(\text{Mod } A)$ :

$$\begin{aligned} & R^b \text{Hom}_B(R^b \text{Hom}_A(A_A, \text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)), \text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)) \\ & \cong \text{Hom}_B(\text{Hom}_R(T_B^\bullet, \omega^\bullet), \text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)) \\ & \cong \text{Hom}_R({}_A T_B^\bullet \overset{\circ}{\otimes}_B \text{Hom}_R(T_B^\bullet, \omega^\bullet), \omega^\bullet) \\ & \cong \text{Hom}_R(\text{Hom}_R \text{Hom}_B({}_A T_B^\bullet, T_B^\bullet), \omega^\bullet). \end{aligned}$$

Since  ${}_A T_B^\bullet$  is a tilting  $A$ - $B$ -bimodule complex, the natural morphism  $A_A \rightarrow \text{Hom}_R({}_A T_B^\bullet, T_B^\bullet)$  is a quasi-isomorphism in  $K(\text{Mod } A)$ . By the duality of  $\omega^\bullet$ , we have the following isomorphisms:

$$\begin{aligned} \text{Hom}_R(\text{Hom}_R(\text{Hom}_B({}_A T_B^\bullet, T_B^\bullet), \omega^\bullet), \omega^\bullet) & \cong \text{Hom}_R(\text{Hom}_R(A_A, \omega^\bullet), \omega^\bullet) \\ & \cong A_A. \end{aligned}$$

Similarly,  $\text{Hom}_R({}_A T_B^\bullet, \omega^\bullet)$  satisfies the conditions (C3l), (C4l).  $\square$

We get the non-commutative ring version of results of Grothendieck and Hartsorne [4, Ch. 4, Proposition 2.4].

**Corollary 2.15.** *Let  $R$  be a commutative Noetherian ring.  $A$  an  $R$ -algebra which is finitely generated as an  $R$ -module. If  $\omega^\bullet$  is a dualizing  $R$ -module complex, then  $\text{Hom}_R(A, \omega^\bullet)$  is a dualizing  $A$ -bimodule complex.*

### 3. A Morita duality theorem for derived categories

Let  $k$  be a commutative ring. We call a  $k$ -algebra  $A$  a projective  $k$ -algebra if  $A$  is projective as a  $k$ -module. Let  $A, B$  and  $C$  be projective  $k$ -algebras. According to [3], a projective (resp., injective)  $B^{\text{op}} \otimes_k A$ -module is projective (resp., injective) as both a right  $A$ -module and a left  $B$ -module. According to [2, 11, 13] we have the following derived functors:

$$\begin{aligned} R\text{Hom}_A^\bullet(-, -) : D(\text{Mod } B^{\text{op}} \otimes_k A)^{\text{op}} \times D(\text{Mod } C^{\text{op}} \otimes_k A) &\rightarrow D(\text{Mod } C^{\text{op}} \otimes_k B), \\ - \otimes_A^L - : D(\text{Mod } B^{\text{op}} \otimes_k A) \times D(\text{Mod } A^{\text{op}} \otimes_k C) &\rightarrow D(\text{Mod } B^{\text{op}} \otimes_k C). \end{aligned}$$

Let  $D^{\text{b}}(\text{Mod } A)_{\text{fid}}$  be the triangulated subcategory of  $D^{\text{b}}(\text{Mod } A)$  generated by complexes which are isomorphic to complexes in  $K^{\text{b}}(\text{Inj } A)$ .

**Lemma 3.1.** *Let  $A$  be a ring, and  $rI(A)$  the set of right ideals of  $A$ . For a complex  $X^\bullet \in D^{\text{b}}(\text{Mod } A)$ , the following hold:*

(a) *If there exist an integer  $n$  such that  $\text{Hom}_{D^{\text{b}}(\text{Mod } A)}(\bigoplus_{I \in rI(A)} A/I, X^\bullet[i]) = 0$  for all  $i > n$ , then  $X^\bullet$  belongs to  $D^{\text{b}}(\text{Mod } A)_{\text{fid}}$ .*

(b) *In case of  $A$  being a right Artinian ring, if there exist an integer  $n$  such that  $\text{Hom}_{D^{\text{b}}(\text{Mod } A)}(A/\text{rad } A, X^\bullet[i]) = 0$  for all  $i > n$ , then  $X^\bullet$  belongs to  $D^{\text{b}}(\text{Mod } A)_{\text{fid}}$ .*

**Proof.** (a) By Baer condition. (b) By [1].  $\square$

**Lemma 3.2.** *Let  $A$  be a right coherent projective  $k$ -algebra,  $B$  a left coherent projective  $k$ -algebra. Let  ${}_B V_A^\bullet$  be a  $B$ - $A$ -bimodule complex which belongs to  $D^{\text{b}}(\text{Mod } A)_{\text{fid}}$  as a right  $A$ -module complex, and belongs to  $D^{\text{b}}(B\text{-Mod})_{\text{fid}}$  as a left  $B$ -module complex. Then there exists a bounded  $B$ - $A$ -bimodule complex  ${}_B U_A^\bullet$ , which belongs to  $K^{\text{b}}(\text{Inj } A)$  as a right  $A$ -module complex, and belongs to  $K^{\text{b}}(B\text{-Inj})$  as a left  $B$ -module complex, such that  ${}_B U_A^\bullet$  is isomorphic to  ${}_B V_A^\bullet$  in  $D(\text{Mod } B^{\text{op}} \otimes_k A)$ .*

**Proof.** See [13, Proposition 2.4].  $\square$

By the above lemma, we can replace the conditions (C2r) and (C2l) of cotilting bimodule complexes by the following conditions:

- (C2'r)  ${}_B U_A^\bullet$  belongs to  $D^{\text{b}}(\text{Mod } A)_{\text{fid}}$  as a right  $A$ -module complex;
- (C2'l)  ${}_B U_A^\bullet$  belongs to  $D^{\text{b}}(B\text{-Mod})_{\text{fid}}$  as a left  $B$ -module complex.

**Theorem 3.3.** *Let  $A$  be a right coherent projective  $k$ -algebra and  $B$  a left Noetherian projective  $k$ -algebra. The following are equivalent:*

- (a)  *$A$  is a left strong-Morita derived dual of  $B$ .*
- (b) *There exists a cotilting  $B$ - $A$ -bimodule complex  ${}_B U_A^\bullet$ .*

**Proof.** (b)  $\Rightarrow$  (a): By Corollary 2.8.

(a)  $\Rightarrow$  (b): Let  $F : D(\text{Mod } A) \rightarrow D(B\text{-Mod})$  and  $G' : D(B\text{-Mod}) \rightarrow D(\text{Mod } A)$  be continuous  $\partial$ -functors satisfying that  $A$  is a left strong-Morita derived dual of  $B$ . By Proposition 2.1, we can take a right adjoint pair  $\{F : D(\text{Mod } A) \rightarrow D(B\text{-Mod}), G : D(B\text{-Mod}) \rightarrow D(\text{Mod } A)\}$  satisfying that  $A$  is a left strong-Morita derived dual of  $B$ . Let  $X^\bullet$  be a complex  $GB \in D_{\text{mod } A}^b(\text{Mod } A)$ . Then we have the following isomorphisms:

$$\begin{aligned} \text{Hom}_{D(\text{Mod } A)}(X^\bullet, X^\bullet[i]) &= \text{Hom}_{D(\text{Mod } A)}(GB, GB[i]) \\ &\cong \text{Hom}_{D(B\text{-mod})}(B, B[i]) \\ &= 0 \quad \text{for all } i \neq 0. \end{aligned}$$

According to [5], there exists a  $B$ - $A$ -bimodule complex  ${}_B U_A^\bullet \in K^-(\text{Proj } B^{\text{op}} \otimes_k A)$  such that  ${}_B U_A^\bullet$  is isomorphic to  $X_A^\bullet$  in  $D(\text{Mod } A)$ , and that the natural left multiplication morphism  $B \rightarrow \text{Hom}_{D(\text{Mod } A)}({}_B U_A^\bullet, {}_B U_A^\bullet)$  is a ring isomorphism. Then  ${}_B U_A^\bullet$  satisfies the conditions (C3r) and (C4r). Since  $B \cong \text{End}_{D(\text{Mod } A)}({}_B U_A^\bullet) \cong \text{End}_{D(\text{Mod } A)}(GB)$ , we have the following isomorphisms as  $B$ - $A$ -bimodules:

$$\begin{aligned} H^i({}_B U_A^\bullet) &\cong \text{Hom}_{D(\text{Mod } A)}(A, {}_B U_A^\bullet[i]) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(A, GB[i]) \\ &\cong \text{Hom}_{D(B\text{-Mod})}(B, FA[i]). \end{aligned}$$

Since  $FA$  belongs to  $D_{B\text{-Mod}}^b(B\text{-Mod})$ , then  ${}_B U_A^\bullet$  belongs to  $D_{B\text{-mod}}^b(B\text{-Mod})$ . Therefore  ${}_B U_A^\bullet$  satisfies the condition (C1). Since  $\bigoplus_{I \in I(A)} A/I$  belongs to  $D^d(\text{Mod } A)$ ,  $F(\bigoplus_{I \in I(A)} A/I)$  belongs to  $D^b(B\text{-Mod})$ . Then there exists an integer  $n$  such that we have

$$\begin{aligned} \text{Hom}_{D(\text{Mod } A)}\left(\bigoplus_{I \in I(A)} A/I, {}_B U_A^\bullet[i]\right) &\cong \text{Hom}_{D(\text{Mod } A)}\left(\bigoplus_{I \in I(A)} A/I, GB[i]\right) \\ &\cong \text{Hom}_{D(B\text{-Mod})}\left(B, F\left(\bigoplus_{I \in I(A)} A/I\right)[i]\right) \\ &= 0 \quad \text{for all } i > n. \end{aligned}$$

By Lemma 3.1(a), we get  ${}_B U_A^\bullet \in D_{\text{mod } A}^b(\text{Mod } A)_{\text{fid}}$ . According to Lemma 3.5, for every complex  $P^\bullet \in K^-({}_B \mathcal{P})$ , we have an isomorphism  $GP^\bullet \cong \mathbf{R} \text{Hom}_B^\bullet(P^\bullet, {}_B U_A^\bullet)$  in  $D(\text{Mod } A)$ . Since  $B$  is left Noetherian, by the continuity of  $G$ , we have

$$\begin{aligned} \mathbf{R} \text{Hom}_B^\bullet\left(\bigoplus_{J \in I(B)} B/J, {}_B U_A^\bullet\right) &\cong \prod_{J \in I(B)} \mathbf{R} \text{Hom}_B^\bullet(B/J, {}_B U_A^\bullet) \\ &\cong \prod_{J \in I(B)} G(B/J) \\ &= G\left(\bigoplus_{J \in I(B)} B/J\right). \end{aligned}$$

Then  $\mathbf{R} \operatorname{Hom}_B^{\bullet}(\bigoplus_{J \in \mathcal{I}(B)} B/J, {}_B U_A^{\bullet})$  belongs to  $D^b(\operatorname{Mod} A)$ . Since  $\{\mathbf{R} \operatorname{Hom}_A^{\bullet}(-, {}_B U_A^{\bullet}): D(\operatorname{Mod} A) \rightarrow D(B\text{-Mod}), \mathbf{R} \operatorname{Hom}_B^{\bullet}(-, {}_B U_A^{\bullet}): D(B\text{-Mod}) \rightarrow D(\operatorname{Mod} A)\}$  is a right adjoint pair, there exists an integer  $n$  such that we have

$$\begin{aligned} & \operatorname{Hom}_{D(B\text{-mod})} \left( \bigoplus_{J \in \mathcal{I}(B)} B/J, {}_B U_A^{\bullet}[i] \right) \\ & \cong \operatorname{Hom}_{D(B\text{-Mod})} \left( \bigoplus_{J \in \mathcal{I}(B)} B/J, \mathbf{R} \operatorname{Hom}_A^{\bullet}(A_A, {}_B U_A^{\bullet})[i] \right) \\ & \cong \operatorname{Hom}_{D(\operatorname{Mod} A)} \left( A_A, \mathbf{R} \operatorname{Hom}_B^{\bullet} \left( \bigoplus_{J \in \mathcal{I}(B)} B/J, {}_B U_A^{\bullet} \right)[i] \right) \\ & = 0 \quad \text{for all } i > n. \end{aligned}$$

By Lemma 3.1(a), we get  ${}_B U_A^{\bullet} \in D_{B\text{-mod}}^b(B\text{-Mod})_{\text{fid}}$ . Since the natural morphism  $B \rightarrow \mathbf{R} \operatorname{Hom}_A^{\bullet}({}_B U_A^{\bullet}, {}_B U_A^{\bullet})$  is an isomorphism in  $D^b(\operatorname{Mod} B^{\text{op}} \otimes_k B)$ , we have an isomorphism  $P \rightarrow \mathbf{R} \operatorname{Hom}_B^{\bullet}(\mathbf{R} \operatorname{Hom}_A(P, {}_B U_A^{\bullet}), {}_B U_A^{\bullet})$ , for every finitely generated projective left  $B$ -module. Then we have an isomorphism  $P^{\bullet} \rightarrow \mathbf{R} \operatorname{Hom}_A^{\bullet}(\mathbf{R} \operatorname{Hom}_B^{\bullet}(P^{\bullet}, {}_B U_A^{\bullet}), {}_B U_A^{\bullet})$  for every  $P^{\bullet} \in K^b({}_B \mathcal{P})$ . By the duality, there exists a complex  $Q^{\bullet} \in K^-({}_B \mathcal{P})$  such that  $A_A \cong GQ^{\bullet}$  in  $D(\operatorname{Mod} A)$ . According to Lemma 3.5, we get an isomorphism  $GQ^{\bullet} \cong \mathbf{R} \operatorname{Hom}_B^{\bullet}(Q^{\bullet}, {}_B U_A^{\bullet})$  in  $D(\operatorname{Mod} A)$ . Since  $\mathbf{R} \operatorname{Hom}_B^{\bullet}(Q^{\bullet}, {}_B U_A^{\bullet})$  is a  $U_A^{\bullet}$ -limit complex with a sequence  $\{\mathbf{R} \operatorname{Hom}_B^{\bullet}(\tau_{\geq -n} Q^{\bullet}, {}_B U_A^{\bullet})\}$ , we have the following isomorphism in  $D(B\text{-Mod})$ :

$$\begin{array}{ccc} \operatorname{hlim}_{n \rightarrow \infty} \tau_{\geq -n} Q^{\bullet} \rightarrow \operatorname{hlim}_{n \rightarrow \infty} \mathbf{R} \operatorname{Hom}_A^{\bullet}(\mathbf{R} \operatorname{Hom}_B^{\bullet}(\tau_{\geq -n} Q^{\bullet}, {}_B U_A^{\bullet}), {}_B U_A^{\bullet}) & & \downarrow \\ \parallel & \mathbf{R} \operatorname{Hom}_A^{\bullet} \left( \operatorname{hlim}_{\infty \leftarrow n} \mathbf{R} \operatorname{Hom}_B^{\bullet}(\tau_{\geq -n} Q^{\bullet}, {}_B U_A^{\bullet}), {}_B U_A^{\bullet} \right) & \downarrow \\ \operatorname{hlim}_{n \rightarrow \infty} \tau_{\geq -n} Q^{\bullet} \rightarrow \mathbf{R} \operatorname{Hom}_A^{\bullet} \left( \mathbf{R} \operatorname{Hom}_B^{\bullet} \left( \operatorname{hlim}_{n \rightarrow \infty} \tau_{\geq -n} Q^{\bullet}, {}_B U_A^{\bullet} \right), {}_B U_A^{\bullet} \right) & & \downarrow \\ \downarrow & \mathbf{R} \operatorname{Hom}_A^{\bullet}(\mathbf{R} \operatorname{Hom}_B^{\bullet}(Q^{\bullet}, {}_B U_A^{\bullet}), {}_B U_A^{\bullet}) & \downarrow \\ Q^{\bullet} \longrightarrow \mathbf{R} \operatorname{Hom}_A^{\bullet}(\mathbf{R} \operatorname{Hom}_B^{\bullet}(Q^{\bullet}, {}_B U_A^{\bullet}), {}_B U_A^{\bullet}) & & \end{array}$$

where the vertical arrows are isomorphisms in  $D(B\text{-Mod})$ . Then, by the dualities and the property of right adjoint pairs,  $\mathbf{R} \operatorname{Hom}_B^{\bullet}(Q^{\bullet}, {}_B U_A^{\bullet}) \rightarrow \mathbf{R} \operatorname{Hom}_B^{\bullet}(\mathbf{R} \operatorname{Hom}_A^{\bullet}(\mathbf{R} \operatorname{Hom}_B^{\bullet}(Q^{\bullet}, {}_B U_A^{\bullet}), {}_B U_A^{\bullet}), {}_B U_A^{\bullet})$  is an isomorphism in  $D(\operatorname{Mod} A)$ , and hence  $A_A \rightarrow \mathbf{R} \operatorname{Hom}_B^{\bullet}(\mathbf{R} \operatorname{Hom}_A^{\bullet}(A_A, {}_B U_A^{\bullet}), {}_B U_A^{\bullet})$  is an isomorphism in  $D(\operatorname{Mod} A)$ . This implies that  ${}_B U_A^{\bullet}$  satisfies the conditions (C3!) and (C4!).  $\square$

**Lemma 3.4.** Let  $U_A^\bullet$  be a complex in  $D^b(\text{Mod } A)$  which satisfies the condition (C3r), and  $X^\bullet$  a  $U_A^\bullet$ -limit complex with  $(\{X_i^\bullet\}_{i \geq 0}; 0)$ . Then we have  $\text{Hom}_{D(\text{Mod } A)}(X_k^\bullet, U_A^\bullet[l]) = 0$  for all  $l < -k$ .

**Lemma 3.5.** In the situation of the proof in Theorem 3.3, for every complex  $P^\bullet \in K^-(\mathcal{B}\mathcal{P})$ , we have an isomorphism  $GP^\bullet \cong \mathbf{R} \text{Hom}_{\mathcal{B}}^\bullet(P^\bullet, {}_B U_A^\bullet)$  in  $D(\text{Mod } A)$ .

**Proof.** Let  $H := \mathbf{R} \text{Hom}_{\mathcal{B}}^\bullet(-, {}_B U_A^\bullet)$ ,  $P^\bullet$  a complex  $\dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0 \rightarrow \dots$  which belongs to  $K^-(\mathcal{B}\mathcal{P})$ , and  $P_i^\bullet := \tau_{\geq -i} P^\bullet$ . Then we have the following sequences of distinguished triangles:

$$\begin{aligned} GP^{-1}[-1] &\rightarrow GP_1^\bullet \rightarrow GP^0 \rightarrow, & HP^{-1}[-1] &\rightarrow HP_1^\bullet \rightarrow HP^0 \rightarrow, \\ GP^{-2}[-2] &\rightarrow GP_2^\bullet \rightarrow GP_1^\bullet \rightarrow, & HP^{-2}[-2] &\rightarrow HP_2^\bullet \rightarrow HP_1^\bullet \rightarrow, \\ \dots & & \dots & \\ GP^{-i}[-i] &\rightarrow GP_i^\bullet \rightarrow GP_{i+1}^\bullet \rightarrow, & HP^{-i}[-i] &\rightarrow HP_i^\bullet \rightarrow HP_{i+1}^\bullet \rightarrow, \\ \dots, & & \dots & \end{aligned}$$

By inductive step, we construct isomorphisms between distinguished triangles:

$$\begin{array}{ccccccc} GP^{-i}[i] & \xrightarrow{z_i} & GP_i^\bullet & \xrightarrow{y_i} & GP_{i-1}^\bullet & \xrightarrow{x_i} & GP^{-i}[-i+1] \\ \downarrow \beta_i & & \downarrow \alpha_i & & \downarrow \alpha_{i-1} & & \downarrow \beta_i[1] \\ HP^{-i}[-i] & \xrightarrow{w_i} & HP_i^\bullet & \xrightarrow{v_i} & HP_{i-1}^\bullet & \xrightarrow{u_i} & HP^{-i}[-i+1], \end{array}$$

where  $P_0^\bullet := P^0$ . Since the natural morphisms  $B \cong \text{Hom}_{D(\mathcal{B}\text{-Mod})}(GB, GB) \cong \text{Hom}_{D(\mathcal{B}\text{-Mod})}({}_B U_A^\bullet, {}_B U_A^\bullet)$  are isomorphisms, the isomorphism  $GB \cong \mathbf{R} \text{Hom}_{\mathcal{B}}^\bullet(B, {}_B U_A^\bullet)$  induces the isomorphism  $\text{Hom}_{D(\mathcal{B}\text{-Mod})}(GB, GB) \cong \text{Hom}_{D(\mathcal{B}\text{-Mod})}(\mathbf{R} \text{Hom}_{\mathcal{B}}^\bullet(B, {}_B U_A^\bullet), \mathbf{R} \text{Hom}_{\mathcal{B}}^\bullet(B, {}_B U_A^\bullet))$ . Since all  $GP^{-i}$  and all  $HP^{-i}$  belong to  $\text{add } U_A^\bullet$ , we can choose isomorphisms  $\alpha_0$  and  $\beta_1$ , and therefore we can choose an isomorphism  $\alpha_1$ . Assume we have isomorphisms  $\alpha_{i-1}$ ,  $\alpha_i$  and  $\beta_i$  which satisfy the above condition. We also can choose an isomorphism  $\beta_{i+1}: GP^{-i-1}[-i-1] \rightarrow HP^{-i-1}[-i-1]$  such that  $\beta_{i+1}[1] \circ x_{i+1} \circ z_i = u_{i+1} \circ w_i \circ \beta_i$ . Since  $(\beta_{i+1}[1] \circ r_{i+1} - u_{i+1} \circ w_i) \circ z_i = \beta_{i+1}[1] \circ x_{i+1} \circ z_i - u_{i+1} \circ w_i \circ \beta_i = 0$ , by the property of distinguished triangles, there exists a morphism  $s: GP_{i-1}^\bullet \rightarrow HP_{i-1}^\bullet$  such that  $s \circ y_i = \beta_{i+1}[1] \circ x_{i+1} - u_{i+1} \circ w_i$ . But, by Lemma 3.4,  $\text{Hom}_{D(\text{Mod } A)}(GP_{i-1}^\bullet, HP_{i-1}^\bullet[-i]) = 0$ . Therefore  $\beta_{i+1}[1] \circ x_{i+1} = u_{i+1} \circ w_i$ , and hence we also can choose  $\alpha_{i+1}: GP_{i+1}^\bullet \rightarrow HP_{i+1}^\bullet$  which satisfies the above condition. Since  $G$  and  $H$  are contravariant continuous  $\partial$ -functors, we have the following isomorphisms in  $D(\text{Mod } A)$ :

$$GP^\bullet \cong G \varinjlim_{i \rightarrow \infty} P_i^\bullet \cong \varprojlim_{\infty \leftarrow i} GP_i^\bullet \cong \varprojlim_{\infty \leftarrow i} HP_i^\bullet \cong H \varinjlim_{i \rightarrow \infty} P_i^\bullet \cong HP^\bullet. \quad \square$$

**Remark.** Conditions (D2r) and (D2l) are closely related to the property of finite injective dimension of complexes. Indeed, let  $R$  be a commutative Noetherian regular ring of infinite Krull dimension, and  $A := R[X]/(X^2 - a)$ , where  $a$  is a non-zero element in  $N^2$  for some maximal ideal  $N$  of  $R$ . Then  $A$  is a commutative locally Gorenstein ring of infinite Krull dimension which is non-regular. The bimodule  $A$  is a pointwise dualizing complex, but is not a dualizing complex. Moreover,  $A$  induces a duality  $D_{\text{mod } A}^b(\text{Mod } A) \rightarrow D_{\text{mod } A}^b(\text{Mod } A)$  (oral communication with Yoshino). I do not know if an arbitrary locally Gorenstein ring  $A$  induces a self-duality on  $D_{\text{mod } A}^b(\text{Mod } A)$ , or equivalently if for every prime ideal  $P$  of an arbitrary locally Gorenstein ring  $A$ , there is some integer  $n$  such that  $\text{Ext}_A^i(A/P, A) = 0$  for all  $i > n$ . In case of Artinian rings, we can delete the conditions (D2r) and (D2l).

**Corollary 3.6.** *Let  $A$  be a right Artinian projective  $k$ -algebra,  $B$  a left Artinian projective  $k$ -algebra. Then the following are equivalent:*

- (a)  $A$  is a left Morita derived dual of  $B$ .
- (b)  $A$  is a left strong-Morita derived dual of  $B$ .
- (c) There exists a cotilting  $B$ - $A$ -bimodule complex  ${}_B U_A^\bullet$ .

**Proof.** By Theorem 3.3, it remains to show that (a) implies (c). By the proof of Theorem 3.3, suffices to show that  ${}_B U_A^\bullet$  belongs to  $D_{\text{mod } A}^b(\text{Mod } A)_{\text{fid}}$  and  $D_{B\text{-mod}}^-(B\text{-Mod})_{\text{fid}}$ . Since  $A$  is right Artinian and  $B$  is left Artinian, in the proof of Theorem 3.3, we can replace  $\bigoplus_{I \in \mathcal{I}(A)} A/I$  and  $\bigoplus_{J \in \mathcal{I}(B)} B/J$  by  $A/\text{rad } A$  and  $B/\text{rad } B$ , respectively. We are done by Lemma 3.1(b).  $\square$

We get a non-commutative ring version of results of Grothendieck and Hartshorne [4, Ch. 5, Theorem 3.1] or Yekutieli [13, Theorem 3.9].

**Proposition 3.7.** *Let  $A$  be a local right coherent projective  $k$ -algebra,  $B$  a left coherent projective  $k$ -algebra, and  ${}_B U_A^\bullet$  a cotilting  $B$ - $A$ -bimodule complex. Let  ${}_B V_A^\bullet$  be any  $B$ - $A$ -bimodule complex in  $D^+(\text{Mod } B^{\text{op}} \otimes_k A)$ . Then  ${}_B V_A^\bullet$  is a cotilting  $B$ - $A$ -bimodule complex if and only if there exist an invertible  $A$ -bimodule  $L$  and some integer  $n$  such that  ${}_B V_A^\bullet$  is isomorphic to  ${}_B U_A^\bullet \otimes_A L[n]$  in  $D^+(\text{Mod } B^{\text{op}} \otimes_k A)$ .*

**Proof.** Let  $L$  be an invertible  $A$ -bimodule. For an integer  $n$ , let  ${}_B V_A^\bullet := {}_B U_A^\bullet \otimes_A L[n]$ . By adjointness and Lemma 2.2 concerning  $L$ , it is not difficult to see that  ${}_B V_A^\bullet$  satisfies the conditions of a cotilting bimodule complex. Conversely, let  ${}_B V_A^\bullet$  be a cotilting  $B$ - $A$ -bimodule complex. Then  $\text{Hom}_B^\bullet(\text{Hom}_A^\bullet(-, {}_B U_A^\bullet), {}_B V_A^\bullet)$  and  $\text{Hom}_B^\bullet(\text{Hom}_A^\bullet(-, {}_B V_A^\bullet), {}_B U_A^\bullet): D_{\text{mod } A}^b(\text{Mod } A) \rightarrow D_{\text{mod } A}^b(\text{Mod } A)$  are derived equivalences. Since  $D_{\text{mod } A}^b(\text{Mod } A) \cong D^b(\text{mod } A) \cong K^{-b}(\mathcal{P}_A)$ , by Lemma 2.13, we have the following isomorphisms:

$$\begin{aligned} \text{Hom}_A^\bullet(\text{Hom}_B^\bullet(-, {}_B U_A^\bullet), {}_B V_A^\bullet) &\cong - \otimes_A^L \text{Hom}_B^\bullet({}_B U_A^\bullet, {}_B V_A^\bullet), \\ \text{Hom}_A^\bullet(\text{Hom}_B^\bullet(-, {}_B V_A^\bullet), {}_B U_A^\bullet) &\cong - \otimes_A^L \text{Hom}_B^\bullet({}_B V_A^\bullet, {}_B U_A^\bullet). \end{aligned}$$



Let  $M^\bullet$  and  $N^\bullet$  be  $A$ -bimodule complexes  $\text{Hom}_B^\bullet({}_B U_A^\bullet, {}_B V_A^\bullet)$  and  $\text{Hom}_B^\bullet({}_B V_A^\bullet, {}_B U_A^\bullet)$ , respectively. It is clear that  $M^\bullet$  and  $N^\bullet$  are contained in  $D_{\text{mod } A}^b(\text{Mod } A)$ . By the dualities, we have the following isomorphisms in  $D(A\text{-Mod})$ :

$$\begin{aligned} \text{Hom}_{D(B\text{-Mod})}({}_B U_A^\bullet, {}_B V^\bullet[i]) &\cong \text{Hom}_{D(\text{Mod } A)}(\text{Hom}_B^\bullet({}_B V^\bullet, {}_B U_A^\bullet), \\ &\quad \text{Hom}_B^\bullet({}_B U_A^\bullet, {}_B U_A^\bullet)[i]) \\ &\cong \text{Hom}_{D(\text{Mod } A)}(\text{Hom}_B^\bullet({}_B V^\bullet, {}_B U_A^\bullet), {}_A A_A[i]) \\ &\cong H^i R \text{Hom}_A^\bullet(N_A^\bullet, {}_A A_A) \quad \text{for all } i. \end{aligned}$$

Then  $M^\bullet$  belongs to  $D_{A\text{-mod}}^b(A\text{-Mod})$ . Similarly,  $N^\bullet$  belongs to  $D_{A\text{-mod}}^b(A\text{-Mod})$ . Also,  $M^\bullet \overset{L}{\otimes}_A N^\bullet$  and  $N^\bullet \overset{L}{\otimes}_A M^\bullet$  are isomorphic to  $A$  in  $D(\text{Mod } A^{\text{op}} \otimes_k A)$ . Let  $p$  be the largest integer such that  $H^p(M^\bullet) \neq 0$ , and let  $q$  be the largest integer such that  $H^q(N^\bullet) \neq 0$ . Then we have  $H^p(M^\bullet) \otimes_A H^q(N^\bullet) \cong H^{p+q}(M^\bullet \overset{L}{\otimes}_A N^\bullet)$  and  $H^q(N^\bullet) \otimes_A H^p(M^\bullet) \cong H^{p+q}(N^\bullet \overset{L}{\otimes}_A M^\bullet)$ . Let  $X := H^p(M^\bullet)$  and  $Y := H^q(N^\bullet)$ . We consider the surjection  $X \otimes_A Y \rightarrow X/X(\text{rad } A) \otimes_A Y/(\text{rad } A)Y$ . Since  $X$  and  $Y$  are finitely generated  $A$ -modules on both sides,  $X/X(\text{rad } A)$  and  $Y/(\text{rad } A)Y$  are nonzero. By locality of  $A$ ,  $X/X(\text{rad } A) \otimes_A Y/(\text{rad } A)Y$  is non-zero, and  $H^p(M^\bullet) \otimes_A H^p(N^\bullet)$  is non-zero. Similarly,  $H^p(N^\bullet) \otimes_A H^p(M^\bullet)$  is non-zero. Then  $p + q = 0$  and  $H^p(M^\bullet)$  is an invertible  $A$ -bimodule with inverse  $H^q(N^\bullet)$ . Let  $H^{-q}(M^\bullet)$  and  $H^q(N^\bullet)$  be  $L$  and  $L^*$ , respectively. By projectivity of  $L$  and  $L^*$ , we have  $M^\bullet \cong M' \oplus L[q]$  in  $D(\text{Mod } A)$  and  $N^\bullet \cong N' \oplus L^*[-q]$  in  $D(A\text{-Mod})$ . Then we have the following isomorphisms in  $D(\text{Mod } k)$ :

$$\begin{aligned} A &\cong M^\bullet \overset{L}{\otimes}_A N^\bullet \\ &\cong L[q] \overset{L}{\otimes}_A L^*[-q] \oplus L[q] \overset{L}{\otimes}_A N^\bullet \oplus M' \bullet \overset{L}{\otimes}_A L^*[-q] \oplus M' \bullet \overset{L}{\otimes}_A N^\bullet. \end{aligned}$$

Then  $L[q] \overset{L}{\otimes}_A N^\bullet \oplus M' \bullet \overset{L}{\otimes}_A L^*[-q] \oplus M' \bullet \overset{L}{\otimes}_A N^\bullet$  is acyclic, and  $M' \bullet$  and  $N^\bullet$  are acyclic. Therefore  $M^\bullet$  and  $N^\bullet$  are isomorphic to  $L[q]$  and  $L^*[-q]$  in  $D(\text{Mod } A^{\text{op}} \otimes_k A)$ , respectively. Hence we have the following isomorphisms in  $D(\text{Mod } B^{\text{op}} \otimes_k A)$ :

$$\begin{aligned} {}_B V_A^\bullet &\cong \text{Hom}_A^\bullet(\text{Hom}_B^\bullet({}_B U_A^\bullet, {}_B U_A^\bullet), {}_B V_A^\bullet) \\ &\cong {}_B U_A^\bullet \overset{L}{\otimes}_A \text{Hom}_B^\bullet({}_B U_A^\bullet, {}_B V_A^\bullet) \\ &\cong {}_B U_A^\bullet \overset{L}{\otimes}_A L[q]. \quad \square \end{aligned}$$

**Remark.** In Proposition 3.7, we can replace ‘cotilting bimodule complex’ by ‘tilting bimodule complex’ under the condition that  $A$  is a local projective  $k$ -algebra and that  $B$  is a projective  $k$ -algebra.

**Example.** For the uniqueness of the cotilting bimodule complex, we need the condition that  $A$  is a local ring. Indeed, let  $A$  be a finite dimensional  $k$ -algebra over a field

$k$  which has the following quiver with relations:

$$1 \bullet \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} \bullet 2,$$

with  $\alpha\beta\alpha = \beta\alpha\beta = 0$ . Then  $A$ ,  $Ae_1 \otimes_k e_1 A \rightarrow A$  and  $Ae_2 \otimes_k e_2 A \rightarrow A$  are dualizing  $A$ -bimodule complexes, where morphisms are natural multiplications.

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